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# FALLING FILM FLOW ALONG VERTICAL PLATE WITH TEMPERATURE DEPENDENT PROPERTIES

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## ABSTRACT

In this paper the gravity-driven Newtonian, incompressible laminar film flow along vertical wall with temperature dependent physical properties is investigated. For the large temperature differences in the boundary layer when Falkner-Skan transformation is no longer applicable, and for the variable physical properties, the Howarth-Dorodnitsyn type transformation is adopted here to render governing PDE to ordinary differential equations. Although initial value method or integral methods are typically used to solve this type of boundary layer problems, in this study, however, coupled governing equations are solved by nonlinear two-point boundary value problem more easily as a first approach. The results obtained by the current boundary value method are compared quantitatively with those obtained by the initial value problem method. The effects of variable physical properties on velocity and temperature are observed for various temperature difference and Prandtl number. The results predict that the hydrodynamic and thermal boundary layer decrease with the increase of wall temperature. Local skin-friction coefficient and local Nusselt number have been obtained.

**Keywords:** Falling Film, Boundary Value Problem, Similarity Transformation.

## 1. INTRODUCTION

Gravity-driven flow of liquid in thin film along a solid surface is a common phenomenon which occurs in everyday life as well as in a variety of industrial applications, for instance, in various types of heat and mass transfer equipments like coolers, evaporators, trickling filters [1], and in checmical and nuclear reactors [2]. The earliest theoretical consideration of variable thermo-physical properties for free convection is the perturbation analysis of Hara [3] for air. Tataev [4] also investigated the free convection of a gas with variable viscosity. A well-known analysis of the variable fluid property problem for laminar free convection on an isothermal vertical flat plate has been presented by Sparrow and Gregg [5]. Pop et al. [6] on the steady of laminar gravity-driven film flow along a vertical wall for Newtonian fluids, investigated for the effect of injection/suction on the heat transfer, which is based on Falkner–Skan type transformation.

The Boussinesq approximation is typically considered for this type of flow covering the boundary layer equations. However, this approximation is suited well for the case of small temperature differences between the wall and the ambient fluid. There are numerous engineering applications where heat transfer in boundary layers and film flows caused by acceleration often involves large temperature differences. In this case, the Boussinesq approximation is no longer applicable, and a new similarity transformation is essential. The similarity transformation inspired by Howarth-Dorodnitsyn transformation is used here following Andersson et al. [2] since Falkner-Skan type of similarity transformation is no longer applicable for temperature dependent physical properties.

A variety of solution approaches are possible for the solution of the hydrodynamics of gravity-driven film flow problem, among these, initial value problem method is mostly used. Initial-value methods are typically solved by using either the finite-difference or Runge-Kutta method, combined with a shooting technique. It seems that there is no research work, to our knowledge, which solves the fluid flow and heat transfer of falling film flow and heat transfer with variable properties (even with constant properties) by the boundary value solution directly. In this study, thus, coupled governing ordinary differential equations are solved by nonlinear two-point boundary value problem more easily for the solution of the fluid flow and heat transfer of falling film with variable properties. In this case, no corresponding adjoint equations or shooting technique is necessary. The primary effort in this current problem is to introduce an approach for the numerical solution of coupled governing boundary layer equations more easily. The particular attention is given to the comparison of the results of the similar problem conducted by Andersson et al. [2], who solved the problem as an initial value problem method. In addition, other significant results are also discussed. Nonlinear empirical temperature-dependent correlations for the thermo-physical properties of water are used following Andersson et al. [2] for the purpose of comparison.

## 2. MATHEMATICAL FORMULATION AND METHODOLOGY

Consider a laminar, incompressible, two-dimensional liquid film flowing down along a vertical wall under the gravitational force, as shown in Fig. 1 with coordinate system. The film density, the dynamic viscosity, the thermal conductivity and the specific heat depend on the temperature, and are denoted by $\rho$, $\mu$, $\kappa$ and $C_p$, respectively. A free stream velocity U is known by the inviscid equation of motion as $U(x) = (2gx)^{1/2}$, where g is the gravitational acceleration. A uniform flow with zero velocity i.e. $U_{x=0} = 0$, enters the system with a constant temperature $T_0$ and falls vertically down

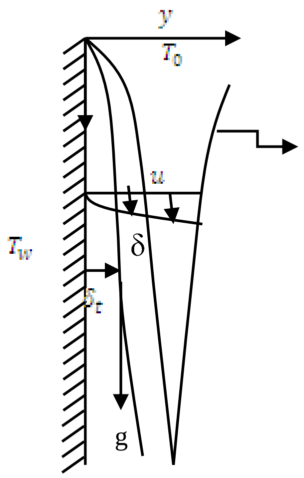



Fig 1. Schematic representation of falling film flow and co-ordinate used

along the smooth wall at a constant temperature $T_w$. A hydrodynamic and a thermal boundary layer are developed along the vertical wall with thicknesses $\delta(x)$ and $\delta_T(x)$, respectively. The conservation equations for mass, momentum and energy equations govern the hydrodynamic and thermal boundary layers. The boundary conditions are prescribed at the wall and at the boundary layer thicknesses. The no slip conditions for the velocities at the wall where temperature is constant at $T_W$. The velocity and temperature tends to the free stream conditions at the boundary layer thicknesses.

For fluids with constant physical properties, the flow system reduces to the problem considered by Andersson [7], a Falkner-Skan type of similarity transformation was enough for the transformation of governing PDEs into a set of ODEs. However, in the present problem, the properties are temperature-dependent, and the Falkner-Skan type of transformation is no longer applicable. Thus, Howarth-Dorodnitsyn transformation [8] is used for the current incompressible fluid flow with variable physical properties to render the problem. One can define a stream function $\psi(x, y)$

$$\rho u = \frac{\partial \psi}{\partial y}, \qquad \rho v = -\frac{\partial \psi}{\partial x}, \tag{1}$$

such that the mass balance is automatically satisfied. The similarity variable $\eta$, and the dependent variables f and $\theta$ for the stream function and temperature, respectively, are defined as,

$$\eta = \left(\frac{3U}{4\upsilon_0 x}\right)^{1/2} \int_0^y \left(\frac{\rho}{\rho_0}\right) dy, \tag{2}$$

$$\psi(x, y) = \rho_0 \left(\frac{4U\upsilon_0 x}{3}\right)^{1/2} f(\eta), \tag{3}$$

$$\theta(x, y) = \frac{T(x, y) - T_0}{T_w - T_0}, \tag{4}$$

where $\rho_0$, $\upsilon_0$ and $T_0$ in equations (2)-(4) is the density, kinematic viscosity and temperature at the inlet of the plate i.e. at x = 0. Upon substituting these variables into the momentum and energy conservation equations, the following two ordinary differential equations are obtained.

$$\left(\frac{\rho\mu}{\rho_0\mu_0} f''\right)' + ff'' + \frac{2}{3}\left(1 - f'^2\right) = 0, \tag{5}$$

$$\left(\frac{\rho\kappa}{\rho_0\kappa_0} \theta'\right)' + \frac{C_p}{C_{p_0}} \Pr_0 f\theta' = 0, \tag{6}$$

where prime in equations (5) and (6) indicate the total differentiation with respect to $\eta$, and $\mu_0$, $C_{p_0}$ and $\kappa_0$ are the viscosity, specific heat and thermal conductivity of the plate at x = 0 and at inflow temperature $T_0$ with $\Pr_0 = \frac{\mu_0 C_{p_0}}{\kappa_0}$. The boundary conditions, in this case, reduces to

$$f(\eta = 0) = f' \quad \text{and} \quad \theta(\eta = 0) = 1, \tag{7a}$$

$$f'(\eta \to \infty) \to 1 \quad \text{and} \quad \theta(\eta \to \infty) \to 0. \tag{7b}$$

The coupled governing ordinary differential equations (5) and (6) with boundary condition (7) constitutes a boundary layer problem.

For given properties and $\Pr_0$, there are five unknown variables in the system to be determined. Note that the properties and $\Pr_0$ cannot be determined readily, rather they are part of the solutions through temperature. Thus, the system is nonlinear and is solved by the coolocation method using MATLAB function 'bvp4c'. Numerical solution is obtained by solving a global system of algebraic equations resulting from the boundary conditions. The adaptive mesh is chosen to make the local error in the tolerance limit, which is taken 1e-6 for all computations.

## 3. RESULTS AND DISCUSSION

In this section, the comparison with other results and the validity of the present solution approach is sought first. The effects for a wide range of flow and temperature parameters on the shear stress and heat transfer are then examined. Water is chosen as working fluid for the purpose of comparison with other relevant works.

It is not possible to solve the present problem with semi-infinite interval, even it is impractical for a very large finite interval. Therefore, the problem is solved upon choosing a finite interval value of $\eta$ which gives a consistent behavior upon further increasing that value. In this case, the solution is extrapolated for one finite value as an initial guess for the new value of the next finite point. Each successive solution is superimposed over those of previous solutions, which is shown in figure 2. It is clear that for $\eta > 2$ the solution gives consistent behavior, however, in this study, $\eta = 4$ is chosen to ensure the solution knowing higher value of $\eta$ is not worthy in comparison to the computation time.

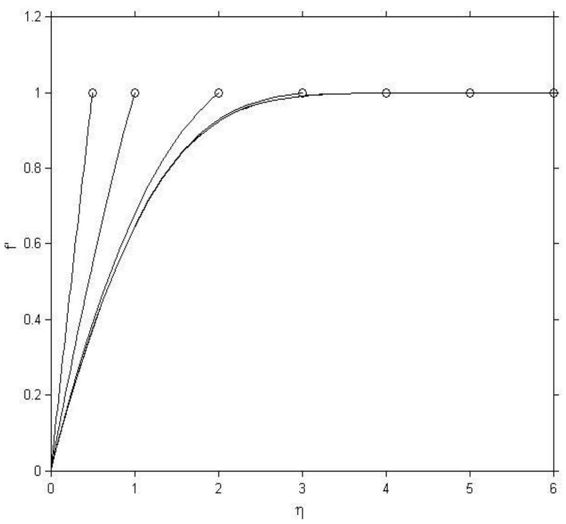



Fig 2. Successive solutions of $f'$ are superimposed to confirm the consistency of the solution's behavior.

Due to the unavailability of numerical data for variable physical properties, the results of the present solution algorithm are compared, at first, to other numerical results [1, 2, 7] for film flow with constant physical properties, which is shown in table 1. In the present formulation, constant physical properties are obtained by setting $\Delta T = 0$. The results clearly give the accuracy up to six decimal points and verify the present method in the range $Pr \in [0.03, 30]$, which however, more likely to be valid for higher Pr values. The local skin friction coefficient and the local Nusselt number with values 0.8997 and 0.4162, respectively, for Pr = 0.7 are also exactly compared with those values predicted numerically for the same Pr in ref. [1] for $\lambda = 0$, where $\lambda$ is the dimensionless temperature difference.

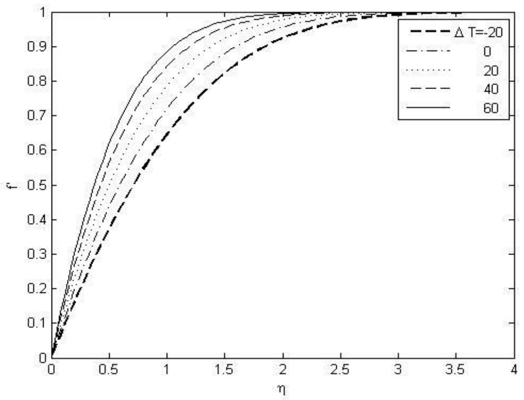
The influence of $\Delta T$ on the velocity and temperature profiles in the hydrodynamic boundary layer in the range $\Delta T \in [-20, 60]$ and for To = $20^o$ C are plotted in figure 3a and figure 3b, respectively for the comparison with those of Andersson et al. [2]. It is evident that both profiles of the current study and of ref. [2] show the similar characteristics with exactly (or very closely) accurate values throughout the domain. It is evident from the figure 3 that both the hydrodynamic and thermal boundary layer shrinks with the increase of $\Delta T$. The adjustment of the fluid velocity from no-slip $f' = 0$ at the wall to the free-stream velocity $f' = 1$ at the edge of the hydrodynamic boundary layer therefore takes place over a gradually narrower region $\delta$. The temperature decreases monotonically with $\eta$, and all the profiles approach asymptotically to a thin region of $\delta_T$. The results predict that for any particular To both the boundary layer thicknesses decrease almost linearly with the increase of $\Delta T$.

Table 1: Comparison of local Nusselt number $Nu_x$ for film flow at constant physical properties with the corresponding results of the references [2, 7]

| Pr | Present | Ref. [2] | Ref. [7] |
|---|---|---|---|
| 0.03 | 0.109105 | 0.109103 | 0.109103 |
| 0.06 | 0.149231 | 0.149232 | 0.149231 |
| 0.10 | 0.186826 | 0.186826 | 0.186825 |
| 0.30 | 0.296863 | 0.296863 | 0.296863 |
| 0.60 | 0.391846 | 0.391846 | 0.391845 |
| 1.0 | 0.477572 | 0.477572 | 0.477572 |
| 3.0 | 0.719274 | 0.719273 | 0.719274 |
| 6.0 | 0.923417 | 0.923417 | 0.923417 |
| 10.0 | 1.106713 | 1.106713 | 1.10671 |
| 30.0 | 1.623386 | 1.623385 | 1.62339 |

Since the properties, notably $\mu$ and $\kappa$, are temperature dependent, the Prandtl number $Pr \equiv \frac{\mu C_p}{\kappa}$ evidently, varies also with temperature, hence it cannot be varied arbitrarily. In the present analysis, however, the effect of the variable Pr is ensured properly through the temperature-dependent $\mu$ and $\kappa$ in the momentum and energy equation, respectively. It should therefore be emphasized that the Prandtl number $Pr_0$ of the incoming flow enters into the transformed governing equations (5) and (6), and governs the flow. The effect of $Pr_0$ on the velocity and temperature profile is then depicted in figure 4a and figure 4b, respectively for the range $Pr_0 \in [4.42, 13.26]$. The corresponding inflow temperature range is $To \in [0, 60]$. Although the ratio of the thermo-physical properties of the equations (5) and (6) will be unity, the effect of $To$ in the equations is ensured via $Pr_0$. The results for the velocity and temperature profiles show qualitatively similar behavior as predicted in figure 3a and figure 3b, respectively for different $\Delta T$. In this case, however, thinner hydrodynamic- and thermal- boundary layers are observed.

(a)




(b)

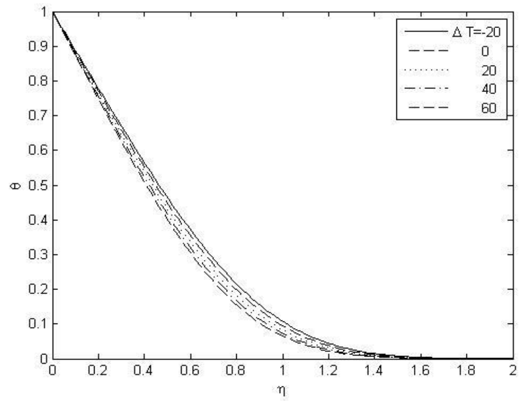



Fig 3. Characteristic (a) velocity profile, $f'(\eta)$ and (b) temperature profile, $\theta(\eta)$, for different $\Delta T$ at $T_0 = 20^o$ C

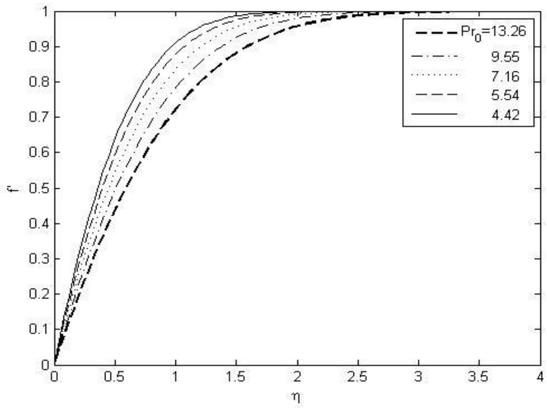
The interplay of inflow Prandtl number and temperature differences on local skin friction coefficient and local Nusselt number in the range $\Delta T \in [-20, 60]$ and $Pr_0 \in [3, 30]$ are depicted in figure 5 and figure 6, respectively. The skin friction coefficient and Nusselt number, representing shear stresses and heat transfer at the wall, related to the gradients as follows

$$C_f = \frac{\tau_w}{\rho_0 U^2} = \frac{\rho_w \mu_w}{\rho_0 \mu_0} . \left(\frac{3}{4}\right)^{1/2} . Re_x^{-1/2} . f''(0) , \qquad (8)$$

$$Nu_x = \frac{x}{T_w - T_0} . \left.\frac{\partial T}{\partial y}\right|_w = -\frac{\rho_w}{\rho_0} . \left(\frac{3}{4}\right)^{1/2} . Re_x^{1/2} . \theta'(0) , \qquad (9)$$

where $Re_x \equiv Ux/\nu_0$ is local Reynolds number and U is the free stream velocity obtained from the inviscid equation of motion. It is clear from figure 5 that the skin friction decreases with the increase of both $Pr_0$ and

(a)




(b)

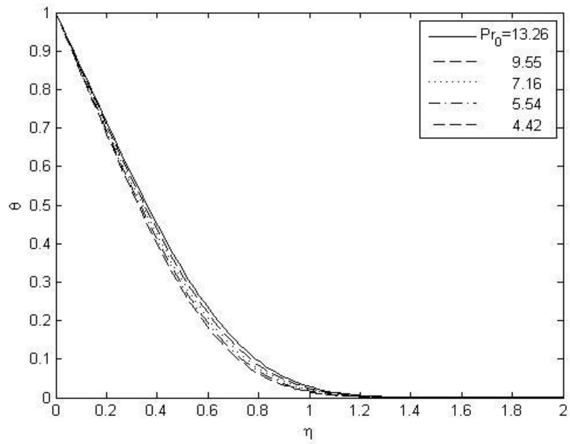



Fig. 4. Characteristic (a) velocity profile $f'(\eta)$, and (b) temperature profile $\theta(\eta)$ for different $Pr_0$ at $T_0=0^o$ C

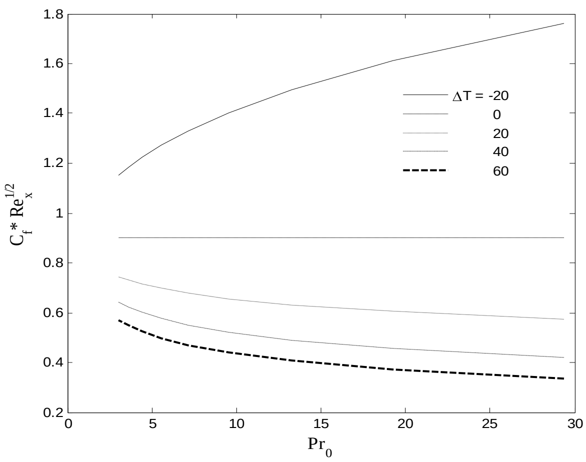



Fig 5. The dependence of local skin friction coefficient on inflow Prandtl number

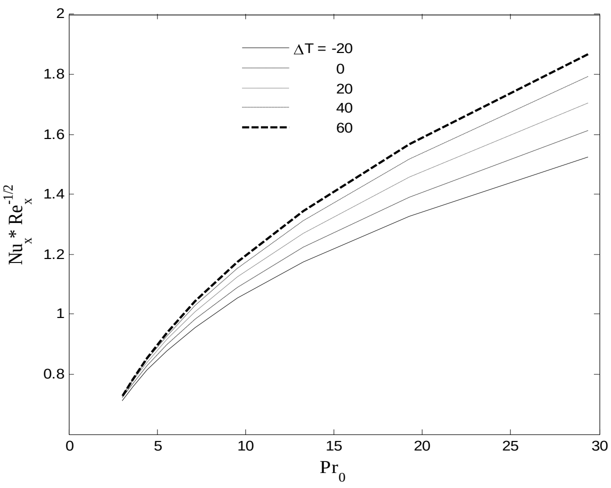



Fig 6. The dependence of local Nusselt number on inflow Prandtl number

$\Delta T$ except at $\Delta T = 0$ where skin friction is always constant since in this case gradients decouples from the temperature. The decreasing tendency of skin friction can be explained by the fact that at any particular $Pr_0$, the increase of $\Delta T$ (which essentially means increase of the wall temperature) reduces the frictional effects on the wall. The similar argument also holds the increase of $Pr_0$ when $\Delta T$ is constant. However, for the wall temperature lower than the inflow temperature, skin friction coefficient shows an opposite tendency with $Pr_0$ where $C_f$ increases with inflow Prandtl number. The variations of local Nusselt number with Prandtl number for different $\Delta T$ are shown in figure 6, where it is observed that Nusselt number, not surprisingly, monotonically increases with both parameters. The increase of Nusselt number with $\Delta T$ is almost linear, where this increase with $\Delta T$, however, is insignificant for smaller $Pr_0$ (or higher $T_0$) values, say $Pr_0 < 5$ (or $T_0 > 35^0$C approx.).

## 4. Conclusion

In this study, a numerical solution is conducted as an alternative to the other available methods for the solution of a laminar, incompressible, two-dimensional liquid film flowing down along a vertical wall under gravity. The MATLAB function 'bvp4c' is used in this purpose, and the solution is achieved by solving a global system of algebraic equations. Adaptive mesh is chosen to minimize the local error. The current results are compared with other relevant works with very good accuracy. The effects of temperature differences and Prandtl number on the fluid flow and heat transfer characteristics are also discussed.

The velocity and thermal boundary layer thicknesses tend to shrink almost linearly with the increase of temperature difference for any particular inflow temperature. Thinner boundary layers are observed for the effect of Prandtl number. In general, the skin friction decreases and Nusselt number increases with the increase of both $Pr_0$ and $\Delta T$. However, for the wall temperature lower than the inflow temperature, skin friction coefficient shows an opposite tendency with Prandtl number.

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## 7. NOMENCLATURE

| Symbol | Meaning | Unit |
|---|---|---|
| $\rho$ | Density | (kg/m$^3$) |
| $\mu$ | Viscosity | (Pa.s) |
| $\kappa$ | Thermal conductivity | (W/(m.K)) |
| $C_p$ | Specific heat | (J/(kg.K)) |
| U | Free stream velocity | (m/s) |
| g | Gravitational acceleration | (m/s$^2$) |
| T | Temperature | ($^o$K) |
| $\delta$ | Hydrodynamic boundary layer thickness | (m) |
| $\delta_t$ | Thermal boundary layer thickness | (m) |
| $\psi$ | Stream function | |
| $Pr_0$ | Inflow Prandtl number | |
| $C_f$ | Skin friction coefficient | |
| $Nu_x$ | Local Nusselt number | |
| $Re_x$ | Local Reynolds number | |